Astronomy Exam 1999
+ Solutions

ASTRONOMY DEPT. PRELIM EXAM PART I: ASTROPHYSICS

Monday June 28, 1999

DIRECTIONS: TIME LIMIT 3 HOURS. Closed book and notes. Answer 8 out of 12 questions. Be sure to write your name on each page. Submit only 8 solutions to be graded.

Equations of Interest

blackbody $B_\nu = (2h\nu^3/c^2)(exp(h\nu/kT) - 1)^{-1}$

Wien's Displacement Law Temp(K) = 0.29 / λ_{max} [cm]

Equation of state for an ideal gas $P = kT\rho/(\mu m_p) = R_g \rho T/\mu$

Maxwell's equations $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial H}{\partial t}$ $\nabla \cdot \mathbf{H} = 0$

$\nabla \times \mathbf{H} = -\frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}\mathbf{j}$ $\nabla \cdot \mathbf{E} = 4\pi\rho$

First law $dQ = dE + Pdv$

Physical constants

speed of light in vacuum $c = 2.998 \times 10^8$ m/s

gravitational constant $G = 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}

Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}$ J s^{-1} m^{-2} K^{-4}

Planck's constant $h = 6.625 \times 10^{-34}$ J s

Elementary (electron) charge $e = 1.60 \times 10^{-19}$ C = 4.80 $\times 10^{-10}$ esu

proton mass $m_p = 1.67 \times 10^{-27}$ kg

Gas constant $R_g = 8.31 \times 10^7$ erg K^{-1} g^{-1} = 8.31×10^3 J K^{-1} kg^{-1}

Boltzmann constant $k = 1.38 \times 10^{-23}$ J/K

electron mass $m_e = 9.11 \times 10^{-31}$ kg

radiation constant $a = 4\sigma/c = 7.56 \times 10^{-16}$ J m^{-3} K^{-4}

Unit conversions:

electron volt= 1.60×10^{-12} erg for incident electrons to de-excite the ion from upper level 2 to lower level 1 is

$$q_{21} = \int_0^{\infty} \sigma_{21} v f(v) dv = \frac{C\bar{\Omega}}{\omega_2 T^{1/2}} \quad \mathrm{cm}^3 \ \mathrm{s}^{-1}$$

where σ_{21} is the de-excitation cross section, $f(v)$ is the Maxwellian distribution of incident electrons, C is a constant, $\bar{\Omega}$ is the collision strength and ω_i is the statistical weight of level i. Write down an expression for the excitation rate coefficient q_{12} in terms of C, $\bar{\Omega}$, the temperature and statistical weights.

(c) Use this expression for q_{12} to derive an expression for the temperature of the HII plasma T as a function of the ion abundance A and other parameters. Assume that the cooling radiation escapes from the plasma without absorption.

(d) How does the plasma temperature depend on distance r from this ionizing star? Explain your answer.

3. A $V = 20$ star observed with LRIS (the Keck imaging spectrograph) produces 1890 detected photo-electrons per second. The R-band sky brightness at Mauna Kea is listed at the CFHT WWW site as 20.9 mag/arcsec^2. The LRIS pixel scale is 0.22 arcseconds/pixel, the readout noise is 8e- and the inverse gain of the system is 2.0 e-/DN.

(a) What is the rate of detected e- from the sky in the R band?

(b) What is the rate of detected e- from a $R = 26$ magnitude star observed at an airmass of 1.2 assuming the extinction coefficient in R is 0.1 mag/(unit airmass)?

(c) Using the symbols defined below, write the expression for signal-to-noise ratio in the brightness measurement of a point-source with a CCD in an aperture r.

(d) Assume that you are measuring all of the light for the $R = 26$ magnitude star in an aperture with a radius of 7 pixels. At what exposure time does the measurement become sky dominated?

(e) For the sky-dominated case, how does the S/N scale with exposure time?

(f) How does the S/N scale with seeing (assume you increase the measuring radius linearly with FWHM of point sources).

R_*	count rate from star, e-/second
R_{sky}	count rate from background, e-/second/pixel
t	exposure time, seconds
r	radius of aperture, pixels

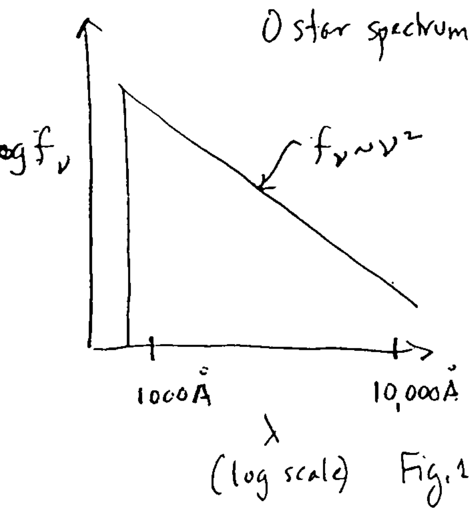

Fig. 1

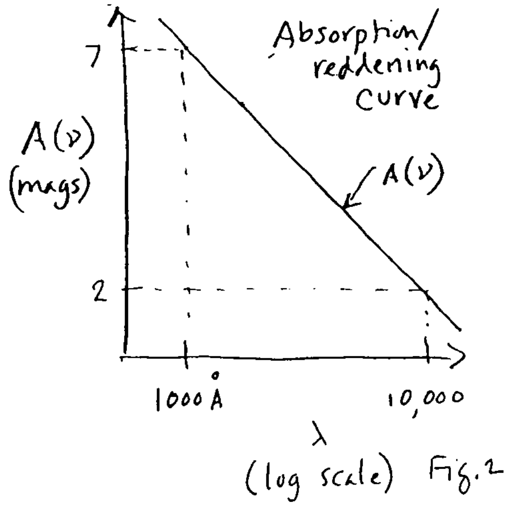

Fig. 2

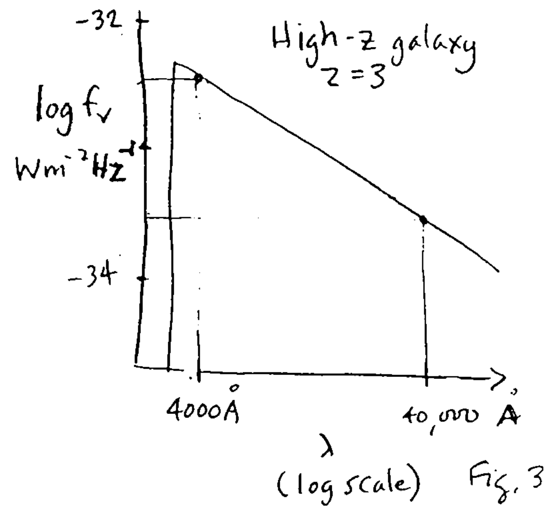

Fig. 3

6. Fig. 1 shows a schematic spectrum of an O star as a Rayleigh-Jeans power law on the long wavelength side, plus a cutoff at the Lyman break (912 Å). Fig. 2 shows an absorption-reddening curve for some hypothetical dust, with arbitrary normalization. Fig. 3 shows the observed spectral energy distribution of a high-redshift galaxy at $z = 3$, as observed on Earth. The high-Z galaxy really consists purely of O stars. Thus it is reddened. Assume that the reddening screen is at the galaxy but lies between us and the stars.

(a) (i) What is the absorption at 4 μ (Earth's frame) in magnitudes? (ii) What is the absorption at 4000 Å(Earth's frame) in magnitudes? (iii) What would the apparent flux be at 4000 Å in the absence of reddening?

(b) Assume that the energy absorbed by dust is reprocessed and emitted as far–IR/submillimeter radiation. You are planning a campaign to observe it. *Very roughly* what is the *total* energy in W m^{-2} that you should expect to detect with a far–IR/submm telescope? (Don't do the integral, just eyeball it).

(c) Now suppose that the dust is *mixed* with the stars or has holes in it. Are your previous absorption estimates too high or too low? Why?

There are many connections between symmetry and conservation of physical quantities.

(b) What quantities are conserved if the Lagrangian is invariant to (i) time translation, (ii) space translation, and (iii) rotation?

(c) What quantities are conserved if a star moves in a central potential V=V(r)? Use the Lagrangian to justify your answer.

(d) Show that if the potential $V = k/r$, the Laplace-Runge-Lenz vector $\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\mathbf{r}/r$ is conserved.

11. In a steady protostellar accretion disk around a 1 solar mass young stellar object, the mass transfer rate is $\dot{M} = 10^{-7} M_\odot \text{ y}^{-1}$, the opacity is $\kappa = \kappa_0 T^2$, and the equation of state is $P = K\rho^{5/3}$.

(a) Derive the density profile in the direction perpendicular to the plane of the disk.

(b) Using an α prescription in which the viscosity $\nu = \alpha \Omega H^2$, derive the surface density distribution as a function of r.

(c) Derive the power index of the continuum black body spectrum from the disk.

12. A uniform-density interstellar cloud has a mass of 10 $M_\odot$ and a radius of 2 $\times 10^{18}$ cm. It is composed of pure molecular hydrogen, has a uniform temperature of 30 K, and is threaded by an uniform magnetic field of 2 μGauss. It is confined by an external medium of density 5×10^{-23} g cm^{-3} and temperature 100 K.

(a) What are the relevant time scales? (give numbers) (b) Is the cloud near virial equilibrium? (c) If not, will the cloud collapse or expand? On what time scale? (d) Will it be able to find an equilibrium state? Assume the evolution is isothermal and with constant magnetic flux.

Astrophysics Exam ... Solutions

Question 1

$$a(r_g) = -Gm(r < r_g)/r_g^2 = -G/r_g^2 \int_0^{r_g} 4\pi r^2 \rho(r) dr$$
$$= -4\pi G\rho_0/(r_g^2 r_0^n) \int_0^{r_g} r^{(2+n)} dr$$
$$= -\frac{4\pi G\rho_0 r_g^{(1+n)}}{r_0^n (3+n)}$$
$$da/dr(r_g) = -\frac{4\pi G\rho_0 r_g^n (1+n)}{r_0^n (3+n)}$$
$$n_{\text{crit}} = -1$$

If $n > n_{\text{crit}}$, the tides are compressive

Question 5

Solution: (a) The Planck mass comes from equating the Compton and the Schwartzchild scales:

$$\frac{\hbar}{M_p c} \approx \frac{GM_p}{c^2}.$$

The uncertainty principle gives the Planck time:

$$M_p c^2 t_{\text{planck}} \sim \hbar \implies t_{\text{planck}} \sim \frac{\hbar}{M_p c^2} \sim \sqrt{G\hbar/c^5}.$$

(b) The horizon size is always $x_H \sim ct$. More precisely,

$$R = R_o \left(\frac{t}{t_o}\right)^{2/3} \text{ and } dr = \frac{cdt}{R} \implies r = 3ct^{1/3} t_o^{2/3} \implies x_H = 3ct.$$

(c) The co-moving event horizon follows from $ds = 0$ in the Robertson Walker metric:

$$\int_0^r dr/c = \int_t^\infty dt e^{-Ht} = H^{-1} e^{-Ht}$$

Therefore the physical size is

$$x = R(t) r = c/H.$$

(a) The ionizing photon luminosity of the star is $\mathcal{L}_*$ s^{-1}.
The maximum (Strömgren) radius that can be ionized is found by balancing the total recombination rate within R_S with $\mathcal{L}_*/h\nu_*$:

$$\frac{\mathcal{L}_*}{h\nu_*} = n_e^2 \alpha_B \frac{4}{3}\pi R_S^3 \quad s^{-1}$$

$$R_S = \left(\frac{3\mathcal{L}_*}{n_e^2 \alpha_B 4\pi h\nu_*}\right)^{1/3}$$

(b) If the de-excitation rate coefficient q_{21} is known, the excitation rate coefficient q_{12} can be found from detailed balancing assuming thermodynamic equilibrium:

$$n_e n_1 q_{12} = n_e n_2 q_{21}$$

but

$$\frac{n_2}{n_1} = \frac{\omega_2}{\omega_1} e^{-E_{12}/kT}$$

therefore

$$q_{12} = q_{21}\frac{n_2}{n_1} = q_{21}\frac{\omega_2}{\omega_1}e^{-E_{12}/kT}$$

$$q_{12} = \frac{C\bar{\Omega}}{\omega_2 T^{1/2}}\frac{\omega_2}{\omega_1}e^{-E_{12}/kT} \quad cm^3/s.$$

Question 2

(c) To find the equilibrium temperature set the heating rate equal to the cooling rate.

Whenever a recombination occurs it is immediately followed by an ionization which delivers energy $(h\nu_* - h\nu_0)$ to the plasma.

$$n_e^2 \alpha_B (h\nu_* - h\nu_0) = n_e \underbrace{(A n_e)}_{n_{ion}} \cdot q_{12} E_{12}$$

$$n_e^2 \frac{a}{T^{1/2}} (h\nu_* - h\nu_0) = n_e^2 A \cdot \frac{c \bar{\Omega}}{\omega_1 T^{1/2}} e^{-E_{12}/kT} E_{12}$$

$$a(h\nu_* - h\nu_0) = \frac{A c \bar{\Omega} E_{12}}{\omega_1} e^{-E_{12}/kT}$$

$$\frac{E_{12}}{kT} = \ln\left[\frac{A c \bar{\Omega} E_{12}}{a(h\nu_* - h\nu_0)\omega_1}\right]$$

$$T = \frac{(E_{12}/k)}{\ln\left[\frac{A c \bar{\Omega} E_{12}}{a(h\nu_* - h\nu_0)\omega_1}\right]}$$

T depends on ion abundance very weakly

(d) T is independent of distance from the star. Although the ionizing flux decreases as $e^{-\tau}/r^2$, the monochromatic spectrum does not vary with r and this is what determines the local plasma temperature.

Question 3

(1) A $R = 20$ star observed with LRIS (the Keck imaging spectrograph) produces 1890 detected photo-electrons per second. The R-band sky brightness at Mauna Kea is listed at the CFHT WWW site as 20.9 mag/arcsec^2. The LRIS pixel scale is 0.22 arcseconds/pixel, the readout noise is 8e- and the inverse gain of the system is 2.0 e-/DN.

(a) What is the rate of detected e-/pixel from the sky in the R band?

$$\left.\begin{aligned} 20 &= c_0 - 2.5 \log(I_{20}) \\ 20.9 &= c_0 - 2.5 \log(I_{20.9}) \end{aligned}\right\} \quad 20.9 - 20 = -2.5 \log\left(\frac{I_{20.9}}{I_{20}}\right)$$

$$I_{20.9} = I_{20} * 10^{\frac{(20.9-20)}{-2.5}} = 1890 * 0.437$$

$$= 825\ e^-/\mathrm{sec}/\boxed{''}$$

$$\text{Area 1 pixel}: (0.22)^2\ \boxed{''} = 0.048\ \boxed{''}$$

$$\left[\text{Sky/pixel} = 825 * 0.048 = 39.9\ e^-/\mathrm{sec}/\boxed{''}\right]$$

(b) What is the rate of detected e- from a $R = 26$ magnitude star observed at an airmass of 1.2 assuming the extinction coefficient in R is 0.1 mag/(unit airmass)?

The star will appear fainter by $\left(0.1 \frac{\mathrm{mag}}{\mathrm{air}}\right)(1.2\ \mathrm{air}) = 0.^m12$

$$R_{\mathrm{apparent}} = 26.^m12$$

$$\left[I_{26.12} = 1890 * 10^{\left(\frac{6.12}{-2.5}\right)} = 6.7\ e^-/\mathrm{sec}\right]$$

(c) Using the symbols defined below in Table 1, write the expression for signal-to-noise ratio in the brightness measurement of a point-source with a CCD in an aperture r.

$$\frac{S}{N} = \frac{R_* t}{\left[R_* t + R_{sky} t \cdot \pi r^2 + RN^2 \cdot \pi r^2\right]^{1/2}}$$

note: assumes negligible dark current and uses fact that $RN \gg \frac{\mathrm{gain}}{2}$

Question 3 - continued

(d) Assume that you are measuring all of the light for the $R = 26$ magnitude star in an aperture with a radius of 7 pixels. At what exposure time does the measurement become sky dominated?

$$\left[R_{sky} \cdot t \cdot \pi (7)^2\right]^{1/2} \gtrsim 3\left[RN^2 \pi (7)^2\right]^{1/2}$$

sky noise $\geq$ 3 times the Readnoise = sky limited for faint sources

$$\left[t \geq 9 \frac{RN^2}{R_{sky}} = 14.4 \text{ seconds}\right]$$

(e) For the sky-dominated case, how does the S/N scale with exposure time.

$$\frac{S}{N} \propto \frac{R_* t}{\left(R_{sky} \cdot t \cdot \pi r^2\right)^{1/2}} \propto t^{1/2}$$

(f) How does the S/N scale with seeing (assume you increase the measuring radius linearly with FWHM of points sources).

$$\frac{S}{N} \propto \frac{R_* t}{\left[R_{sky} \cdot t \cdot \pi r^2 + RN^2 \cdot \pi r^2\right]^{1/2}} \propto \frac{1}{r}$$

R_*	count rate from star	e-/second
R_{sky}	count rate from background	e-/second/pixel
t	exposure time	seconds
r	radius of aperture	pixels
G	inverse-gain	e-/DN
D	dark current	e-/pixel/sec
RN	Readout noise	e-pixel

Question 6

part a)

1) Start by correcting the galaxy observed wavelengths to the galaxy restframe by dividing by $(1+z) = 4$. Notice that Figs. 1 and 2 then cover the same wavelength range as Fig. 3.

The SED of the O star varies as $f_\nu \sim \nu^{+2}$. Observe that the SED of the galaxy varies as $f_\nu \sim \nu^{+1}$ (read from graph). Hence the galaxy is reddened by a factor of 10 (2.5 mag) over a factor of 10 in wavelength, i.e. the color excess

$$E(m_{1000} - m_{10,000}) = 2.5 \text{ mag.}$$

The plotted reddening curve (Fig. 2) varies by 5 mag (factor of 100) over the same range. This is twice the color excess of the galaxy. Hence, the absorption of of galaxy is HALF the plotted reddening curve at each wavelength.

Absorption at 40,000 Å (Earth frame) $= \frac{1}{2} \times 2 \text{ mag} = \underline{1 \text{ mag}}$

2) Absorption at 4,000 Å (Earth frame) $= \frac{1}{2} \times 7 \text{ mag} = \underline{3.5 \text{ mag}}$

3) With reddening, apparent flux at 4000 Å $= 3 \times 10^{-33} \text{ W m}^{-2} \text{Hz}^{-1}$

Absorption (4000 Å) $= 3.5$ mag $\Rightarrow \times 25$

Apparent flux w/o reddening $= 25 \times 3 \times 10^{-33} \text{ W m}^{-2} \text{Hz}^{-1}$

$= \underline{7.5 \times 10^{-32} \text{ W m}^{-2} \text{Hz}^{-1}}$

Question 6

part b)

Flux is absorbed from 4000Å to long wavelengths, but absorption is biggest at blue wavelengths, so take mean $\langle\lambda\rangle = 5000$Å and $\frac{\Delta\lambda}{\lambda} = 0.5$ as band in which absorption occurs.

We see the unabsorbed energy; what is the absorbed energy?
Let total emitted flux be f_{TOT}, observed flux be f_{obs}, absorbed flux be f_{abs}.

$$f_{TOT} = f_{obs} + f_{abs}$$

Now, absorption factor is observed flux / total flux.

$$\frac{f_{obs}}{f_{TOT}} = 10^{-0.4 \times A(v)} = 10^{-0.4 \times 3.0\,\mathrm{mag}} = \frac{1}{16},$$

where $A(v)$ at 5000Å is about 3.0 mag (given that it's 3.5 mags at 4000Å.)

Since $f_{obs} \ll f_{TOT}$,

$$f_{abs} \approx f_{TOT} = 16 \times f_{obs}$$

Hence

$$f_{abs} \approx 16 \times 2 \times 10^{-33}\ \mathrm{W\,m^{-2}\,Hz^{-1}}$$
$$\approx 3 \times 10^{-32}\ \mathrm{W\,m^{-2}\,Hz^{-1}}$$

part b) continued

Total energy absorbed is

$$E = f_{abs} \times \Delta\nu$$

$$\frac{\Delta\nu}{\nu} = \frac{\Delta\lambda}{\lambda} = 0.5$$

$$\Delta\nu = 0.5 \times \nu = 0.5 \times \frac{c}{\lambda} = 0.5 \times \frac{3\times10^{8}\ \mathrm{m/s} \times 10^{10}\ \mathrm{Å/m}}{5000\ \mathrm{Å}}$$

$$= 6\times10^{14}\ \mathrm{Hz}$$

$$E \approx 3\times10^{-32}\ \mathrm{W\,m^{-2}\,Hz^{-1}} \times 6\times10^{14}\ \mathrm{Hz}$$

$$E \approx 1.8\times10^{-17}\ \mathrm{W\,m^{-2}}$$

part c)

ANSWER: If the dust is mixed or has holes in it some of the stars will be less obscured than average. Their flux will dominate the light, & the galaxy will look less reddened. Since the absorptions were based on the color excess, we will have underestimated the absorption.

Question 7

a) The iron group is where a combination of attraction by the strong force (volume term), repulsion by the electrical force, and reduction of binding because of the surface area (surface term) is maximized. Further a nucleus with Z = N has its nucleons in the most tightly bound shells for any combination of A = Z + N nucleons.

Most importantly, the iron-56 ejected by supernovae is made as Ni-56, a double magic nucleus with Z = N = 28. 28 is a closed shell.

b) The iron in the sun has been made by explosive silicon burning as radioactive 56Ni. The iron in a collapsing iron core is made as itself in much more neutron-rich circumstances. In both cases, synthesis requires temperatures above 5 x 10**9 K. Iron is mostly made in a state of nuclear statistical equilibrium. To make 56Ni one needs a small or zero neutron excess (eta =0 or Y_e = 0.5). For more neutron rich conditions the most tightly bound nucleus (i.e., in the collapsing iron core) can be something else - 54Fe, 58Fe, or even 62Ni.

c) Iron is made in supernovae of all types - especially Type II and Ia.

d) The decay of 56Ni and 56Co produces the entire optical display of Type Ia/b/c supernovae. It gives a tail on the light curve of Type IIp

e) Iron is a primary element that can be made in a star regardless

of its initial metallicity. Metallicity is mostly CNO but that is mostly O and the O/Fe ratio in a generation of massive stars should be about the same. Unfortunately the production of Fe in SN Ia clouds this simple relation.

Question 8

Solution: (a) Since the emission is not symmetric in the rest frame, the power radiated is not a Lorentz invariant, so that

$$L = \frac{dE}{dt} = \frac{dE'}{dt'}\frac{\gamma(1+\beta)}{\gamma} = L_o(1+\beta).$$

(b) If no momentum is emitted, then the power radiated is a Lorentz invariant, so that

$$L = L_o$$

(c) Write the transformation as

$$\frac{dE}{d\nu d\Omega dt} = \frac{dE'}{d\nu' d\Omega' dt'}\frac{dt'}{dt}\frac{d\Omega'}{d\Omega}\frac{dE}{dE'}\frac{d\nu'}{d\nu}.$$

Substituting in

$$\frac{dE}{d\nu d\Omega dt} = \frac{L_o}{4\pi}\delta(\nu'-\nu_o)\,\frac{1}{\gamma}\,\frac{1}{\gamma^2(1-\beta\cos\theta)^2}\,\gamma(1-\beta\cos\theta)\,\frac{1}{\gamma(1-\beta\cos\theta)}$$

Use the relation $\nu' = \gamma\nu(1-\beta\cos\theta)$ in the δ function, and integrate over angle to get

$$\frac{dE}{d\nu dt} = \frac{L_o\nu}{2\beta\gamma^2\nu_o^2}$$

where the frequency is restricted to the range

$$\frac{\nu_o}{\gamma(1-\beta)} < \nu < \frac{\nu_o}{\gamma(1+\beta)}$$

Question 1

a) κ is the Rosseland mean opacity

$$\frac{1}{\kappa} = \frac{\int_0^\infty \frac{1}{\kappa_{\omega a}[1-\exp(-\hbar\omega/kT)]+\kappa_{\omega s}} \frac{dB_\omega}{dT}\, d\omega}{\int_0^\infty \frac{dB_\omega}{dT}\, d\omega}$$

where $\kappa_{\omega a}$ is the true absorption coefficient and $\kappa_{\omega s}$ is the scattering coefficient. Note that the true absorption is corrected for induced emission.

b) The principal assumption is Local Thermodynamic Equilibrium (the mean free path of a photon is very short compared to the temperature scale height).

c) $$\frac{dT}{dP} = \frac{3\kappa L_r}{16\pi G a c T^3 M_r}$$

d) $$\frac{T^4}{4} = \frac{3\kappa L_r P}{16\pi G a c M_r}$$

e) the radiative gradient is $\frac{d\ln T}{d\ln P}$

$$4\ln T = \text{const} + \ln P$$

$$\frac{d\ln T}{d\ln P} = \frac{1}{4}$$

the adiabatic gradient is $\frac{\Gamma_2 - 1}{\Gamma_2} = 0.4$ for an ideal gas

$\left(\frac{d\ln T}{d\ln P}\right)_{radiative} < \left(\frac{d\ln T}{d\ln P}\right)_{adiabatic}$ so the envelope is stable against convection.

Question 10

a) Describe the physical bases of a) Newton's description, b) Lagrangian formulation, and c) Hamilton's principle in classical mechanics:

>>a) Newton's description relies on an outside agent (force) acting
>>on bodies and bodies depend only on local forces.
>>b) Lagrangian formation deals with quantities associated with the
>>body such as the kinetic and potential energy.
>>c) Hamilton's principle states that bodies select a path which
>>minimizes its action

There are many connections between symmetry and conservation of physical quantities.
b) What quantities are conserved if the Lagrangian is invariant to i) time translation, ii) space translation, and iii) rotation?

>>i)energy, ii)linear momentum, and iii)angular momentum

c) What quantities are conserved if a star moves in a central potential V=V(r). Use the Lagrangian to justify your answer.

>>L = (rdot**2 + r**2 thetadot**2) - V is invariant of
>>time translation and rotation but not space translation.
>>Therefore energy and angular momentum are conserved.
>>But linear momentum is not conserved.

d) Show that if the potential $V=k/r$, the Laplace-Runge-Lenz vector ${\bf A} = {\bf p} x {\bf L} - m k {\bf r}/r$ is conserved.

$$V = \frac{k}{r} \qquad \nabla V = -\frac{k}{r^3}\underline{r}$$

$$\dot{\underline{P}} = \underline{\nabla} V = -\frac{k}{r^3}\underline{r}$$

Angular momentum $\quad \underline{L} = m\,\underline{r}\times\dot{\underline{r}}$

$$\frac{d\underline{L}}{dt} = m(\dot{\underline{r}}\times\dot{\underline{r}} + \underline{r}\times\ddot{\underline{r}}) = 0$$

$$\frac{d}{dt}(\underline{P}\times\underline{L}) = \frac{d\underline{P}}{dt}\times\underline{L} = -\frac{mk}{r^3}\,\underline{r}\times(\underline{r}\times\dot{\underline{r}})$$

$$= -\frac{mk}{r^3}\left[\underline{r}(\dot{\underline{r}}\cdot\underline{r}) - r^2\dot{\underline{r}}\right]$$

$$= mk\frac{d}{dt}\left(\frac{\underline{r}}{r}\right) = \frac{d}{dt}\left(mk\frac{\underline{r}}{r}\right)$$

$$\therefore \frac{d}{dt}\left(\underline{P}\times\underline{L} - \frac{mk}{r}\underline{r}\right) = 0$$

$$\therefore \underline{P}\times\underline{L} - \frac{mk}{r}\underline{r} = \text{constant}.$$

a) $\frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{GM}{r^3} z = -\Omega^2 z$ pressure balance

$P = K\rho^\gamma \qquad \gamma = 5/3$

$K\gamma \rho^{\gamma-2} \frac{\partial \rho}{\partial z} = -\Omega^2 z.$

$\frac{K\gamma}{(\gamma-1)} \frac{\partial \rho^{\gamma-1}}{\partial z} = -\Omega^2 z$ $\qquad \rho_0 = \rho$ at central plane $z = 0$

$\therefore \rho^{\gamma-1} - \rho_0^{\gamma-1} = -\frac{(\gamma-1)}{K\gamma} \frac{\Omega^2 z^2}{2}$

sound speed

$C_s^2 = \frac{\partial P}{\partial \rho} = \gamma K \rho^{\gamma-1}$

$\therefore \rho = \rho_0 \left[1 - \frac{(\gamma-1)}{K\gamma\rho_0^{\gamma-1}} \frac{\Omega^2 z^2}{2}\right]^{\frac{1}{\gamma-1}}$

$= \rho_0 \left[1 - \left(\frac{\gamma-1}{2}\right) \frac{\Omega^2 z^2}{C_s^2(0)}\right]^{\frac{1}{\gamma-1}}$

ρ vanishes at $z =$

$\rho(z) = \rho_0 \left[1 - \frac{1}{3} \frac{\Omega^2 z^2}{C_s^2(0)}\right]^{3/2} \Rightarrow H = \frac{C_s(0)\sqrt{3}}{\Omega}$

b) In a steady state, continuity equation implies $\dot{m} = 2\pi \Sigma u_r r =$ constant.

$\nu = \alpha \Omega H^2$

momentum eqn $\Rightarrow u_r = -\frac{3}{2}\frac{\nu}{r}$ inflow

$H = \sqrt{3}\frac{C_s}{\Omega}$

$\therefore \dot{m} = -3\pi\Sigma\nu = -9\pi\alpha\Sigma\frac{RT}{\Omega}$ $\qquad \therefore \nu = 3\alpha . \frac{C_s^2}{\Omega} = 3\alpha \frac{RT}{\Omega}$

note $\dot{m}$ is negative for inflow

$\Rightarrow \Sigma = (-\dot{m}\Omega)/9\pi\alpha RT$

Energy equation implies

$\tau \equiv$ opacity $= \frac{1}{2}\Sigma K = \frac{K_0}{2}\Sigma T_c^2$

$\frac{9}{4}\Omega^2 . \Sigma\nu = 2\sigma T_e^4 = \frac{2\sigma T_c^4}{\tau} = \frac{4\sigma T_c^2}{K_0 \Sigma}$

$\Downarrow$

$\left[\frac{3}{16\pi\sigma}\Omega^2(-\dot{m}) K_0 \Sigma\right]^{1/2} = T_c$

$\therefore \Sigma = (-\dot{m}\Omega)/9\pi\alpha R\left[\frac{3}{16\pi\sigma}\Omega^2(-\dot{m}) K_0 \Sigma\right]^{1/2}$

Question ...

$$\Sigma = \left[\frac{-\dot{m}\,16\pi\sigma}{3\,\kappa_0}\right]^{\frac{1}{3}}$$ which is independent of radius !

c) Effective temperature

$$\frac{9}{4}\Sigma\nu\Omega^2 = 2\sigma T_e^4$$

$$\therefore \frac{3}{4\pi}\frac{GM}{r^3}(-\dot{m}) = 2\sigma T_e^4$$

$$\therefore T_e = \frac{3}{8\pi\sigma}(-\dot{m})^{1/4}\Omega^{\frac{1}{2}} \propto r^{-3/4}$$

black body spectrum

$$I_\nu = \frac{2h\nu^3}{c^2(e^{\frac{h\nu}{kT}}-1)}$$

$$F_\nu = 2\pi\int I_\nu\, r\,dr$$

$$= \frac{4\pi h\nu^3}{c^2}\int\frac{r\,dr}{e^{\frac{h\nu}{kT}}-1}$$

Let $x = h\nu/kT_e$ & $T_e = T_1\left(\frac{r}{r_1}\right)^{-3/4}$

$$\therefore F_\nu \propto \nu^{\frac{1}{3}}\int_0^{\infty}\frac{x^{5/3}dx}{e^x-1}$$

$\therefore$ Power index is $\frac{1}{3}$.

Question 12

a) free-fall time $= \left(\frac{3\pi}{32 G\rho}\right)^{1/2} = 8.6 \times 10^{13}$ s

sound crossing time $= R/c_s = 5.66 \times 10^{13}$ s

Alfvén crossing time $= R/v_A = 8.68 \times 10^{13}$ s

b) Virial terms:

gravitational $= -\frac{.6 G M^2}{R} = -8.04 \times 10^{42}$ erg

2x thermal $= 2 \times \frac{3}{2} \frac{R_g}{\mu} T M = 7.48 \times 10^{43}$ erg

magnetic $= \frac{B^2}{8\pi} \times \mathrm{vol} = 5.33 \times 10^{42}$ erg

surface term $= 3 P_{ext} \times \mathrm{vol} = 2.1 \times 10^{43}$ erg

The thermal term dominates and the ratio of internal pressure to external pressure is 3.6. The cloud is not in virial equilibrium (but not too far off).

c) the cloud will expand. A sound wave propagates into the external medium The time scale is the sound-crossing time.

d) If the evolution is isothermal and flux-conserving, the gravitational term scales as $1/R$, the magnetic term scales as $1/R$ ($BR^2 = $ const), the thermal term stays the same, and the surface term increases with the volume. So the cloud will reach equilibrium when the internal pressure approximately matches the external pressure.