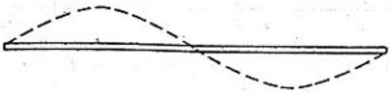
PROBLEM SET #3

9.5 A thin linear antenna of length d is excited in such a way that the sinusoidal current makes a full wavelength of oscillation as shown in the figure.



Problem 9.5

(*a*) Calculate ~~exactly~~ the power radiated per unit solid angle and plot the angular distribution of radiation.
(*b*) Determine the total power radiated

14.2 Using the Liénard-Wiechert fields, discuss the time-average power radiated per unit solid angle in nonrelativistic motion of a particle with charge e, moving
(*a*) along the z axis with instantaneous position $z(t)=a\cos\omega_0 t$,
(*b*) in a circle of radius R in the x-y plane with constant angular frequency ω_0.
Sketch the angular distribution of the radiation and determine the total power radiated in each case.

14.3 A *nonrelativistic* particle of charge ze, mass m, and kinetic energy E makes a *head-on* collision with a fixed central force field of finite range. The interaction is repulsive and described by a potential $V(r)$, which becomes greater than E at close distances.
(*a*) Show that the total energy radiated is given by

$$\Delta W=\frac{4}{3}\frac{z^2e^2}{m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{min}}^{\infty}\left|\frac{dV}{dr}\right|^2\frac{dr}{\sqrt{V(r_{min})-V(r)}}$$

where r_{min} is the closest distance of approach in the collision.
(*b*) If the interaction is a Coulomb potential $V(r)=zZe^2/r$, show that the total energy radiated is

$$\Delta W=\frac{8}{45}\frac{zmv_0^5}{Zc^3}$$

where v_0 is the velocity of the charge at infinity.

14.5 As in Problem 14.2*a* a charge e moves in simple harmonic motion along the z axis, $z(t')=a\cos(\omega_0 t')$.

(*a*) Show that the instantaneous power radiated per unit solid angle is:

$$\frac{dP(t')}{d\Omega}=\frac{e^2c\beta^4}{4\pi a^2}\frac{\sin^2\theta\cos^2(\omega_0 t')}{(1+\beta\cos\theta\sin\omega_0 t')^5}$$

where $\beta=a\omega_0/c$.
(*b*) By performing a time averaging, show that the average power per unit solid angle is:

$$\frac{dP}{d\Omega}=\frac{e^2c\beta^4}{32\pi a^2}\left[\frac{4+\beta^2\cos^2\theta}{(1-\beta^2\cos^2\theta)^{7/2}}\right]\sin^2\theta$$

(c) Make rough sketches of the angular distribution for nonrelativistic and relativistic motion.

14.10 Bohr's correspondence principle states that in the limit of large quantum numbers the classical power radiated in the fundamental is equal to the product of the quantum energy ($\hbar\omega_0$) and the reciprocal mean lifetime of the transition from principal quantum number n to $(n-1)$.
(*a*) Using nonrelativistic approximations, show that in a hydrogen-like atom the transition probability (reciprocal mean lifetime) for a transition from a circular orbit of principal quantum number n to $(n-1)$ is given classically by

$$\frac{1}{\tau}=\frac{2}{3}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{\hbar c}\right)^4\frac{mc^2}{\hbar}\frac{1}{n^5}$$

(*b*) For hydrogen compare the classical value from (*a*) with the correct quantum-mechanical results for the transitions $2p\rightarrow 1s$ (1.6×10^{-9} sec), $4f\rightarrow 3d$ (7.3×10^{-8} sec), $6h\rightarrow 5g$ (6.1×10^{-7} sec).

14.16 Consider the synchrotron radiation from the Crab nebula. Electrons with energies up to 10^{13} eV move in a magnetic field of the order of 10^{-4} gauss.
(*a*) For $E = 10^{13}$ eV, $B = 3\times10^{-4}$ gauss, calculate the orbit radius ρ, the fundamental frequency $\omega_0 = c/\rho$, and the critical frequency ω_c. What is the energy $\hbar\omega_c$ in keV?
(*b*) Show that for a relativistic electron of energy E in a constant magnetic field the power spectrum of synchrotron radiation can be written

$$P(E,\omega) = \text{const}\left(\frac{\omega}{E^2}\right)^{1/3} f\left(\frac{\omega}{\omega_c}\right)$$

where $f(x)$ is a cutoff function having the value unity at $x = 0$ and vanishing rapidly for $x \gg 1$ [e.g., $f \simeq \exp(-2\omega/3\omega_c)$, as in Problem 14.14], and $\omega_c = (eB/mc)(E/mc^2)^2 \cos\theta$, where θ is the pitch angle of the helical path. Cf. Prob. 14.9(*a*).
(*c*) If electrons are distributed in energy according to the spectrum $N(E)\,dE \sim E^{-n}\,dE$, show that the synchrotron radiation has the power spectrum

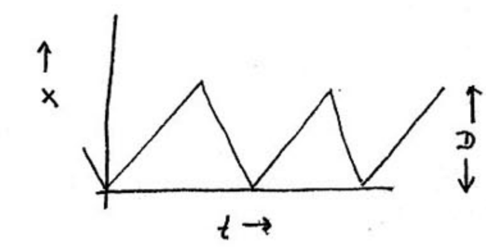
$$\langle P(\omega)\rangle\, d\omega \sim \omega^{-\alpha}\, d\omega$$

where $\alpha = (n-1)/2$.
(*d*) Observations on the radiofrequency and optical continuous spectrum from the Crab nebula show that on the frequency interval from $\omega \sim 10^8\ \text{sec}^{-1}$ to $\omega \sim 6\times10^{15}\ \text{sec}^{-1}$ the constant $\alpha \simeq 0.35$. At frequencies above $10^{18}\ \text{sec}^{-1}$ the spectrum of radiation falls steeply with $\alpha \gtrsim 1.5$. Determine the index n for the electron-energy spectrum, as well as an upper cutoff for that spectrum. Is this cutoff consistent with the numbers of part (*a*)?
(*e*) The half-life of a particle emitting synchrotron radiation is defined as the time taken for it to lose one half of its initial energy. From the result of Problem 14.4(*b*) find a formula for the half-life of an electron in years when B is given in milligauss and E in GeV. What is the half-life using the numbers from part (*a*)? How does this compare with the known lifetime of the Crab nebula? Must the energetic electrons be continually replenished? From what source?

iii.1) A linearly polarized wave with field $\vec{E}_0 = 2\,\hat{e}_x$ and a circularly polarized wave with $|\vec{E}_0| = \sqrt{2}$ move in the z direction. What are the Stokes parameters and polarizations of the resulting wave if (a) they add incoherently, and (b) they add coherently and both waves have the same frequency and maximum E_x at the same time?

iii.2) Derive the linear and circular polarization of cyclotron radiation of electrons with $\beta \ll 1$ and pitch angle $\alpha = \pi/2$ observed from a direction making angle θ with the magnetic field.

iii.3) A non-relativistic electron elastically bounces between two walls with trajectory as shown:



What is the spectrum emitted by this electron in a direction making angle θ with its path?